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# $SU(2)/U(1)$ Dynamical System

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The $SU(2)/U(1)$ dynamical system is defined and its dynamics are studied. It is shown to be chaotic in a certain range of parameters. Links corresponding to the cycles of $SU(2)/U(1)$ are studied.

## 1. INTRODUCTION

Dynamical systems play important roles in theoretical as well as experimental science (Cvitanovic, 1989). Recently (Okninski, 1992) a group-theoretic approach to generate dynamical systems has been discovered and used to study the Euclidean group in two-dimensions $E(2)$. Here we use this approach to derive and study the $SU(2)/U(1)$ dynamical system [$SU(2)/U(1)$ DS]. This system has been proposed (Okninski, 1988) to study scattering problems.

In Section 2 the system is derived and its fixed point and stability are studied. It is shown to be chaotic for a certain range of its parameter. In Section 3 links corresponding to $SU(2)/U(1)$ DS are defined and some of its properties are studied.

## 2. THE MODEL

The Shimizu–Leutbecher sequence $\{G_n\}$ (Shimizu, 1963; Leutbecher, 1967) is defined by

$$G_{n+1} = G_n H G_n^{-1}, \qquad n = 1, 2, 3, \ldots \tag{1}$$

where $G_0 = G$ is a Lie group and $H$ is a subgroup of $G$. Choosing $G$ to be $SU(2)$ and $H = U(1)$, we get

$$G_n = \exp[i(\chi/2)\boldsymbol{\sigma} \cdot \boldsymbol{\omega}_n], \qquad H = \exp[i(\chi/2)\sigma_3] \tag{2}$$

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where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $\boldsymbol{\omega}_n = (x_n, y_n, z_n)$ is a unit vector, i.e.,

$$x_n^2 + y_n^2 + z_n^2 = 1 \tag{3}$$

Using (1) and (2), we get the $SU(2)/U(1)$ DS

$$x_{n+1} = 2bx_n z_n - [b(1-b)]^{1/2} y_n \tag{4a}$$

$$y_{n+1} = 2by_n z_n - [b(1-b)]^{1/2} x_n \tag{4b}$$

$$z_{n+1} = 2bz_n^2 + (1-2b) \tag{4c}$$

where $b = \sin^2(\chi/2)$, i.e., $0 < b < 1$. In deriving the system (4) the following relations of Pauli matrices are useful:

$$\sigma_l \sigma_j - \sigma_j \sigma_l = 2i\pi\varepsilon_{ljk}\sigma_k \tag{5a}$$

$$\sigma_l \sigma_j + \sigma_j \sigma_l = 2\delta_{jl} \mathrm{I} \tag{5b}$$

$$\exp(i\theta\boldsymbol{\omega}_n \cdot \boldsymbol{\sigma}) = \cos\theta + i\boldsymbol{\omega}_n \cdot \boldsymbol{\sigma} \sin\theta \tag{5c}$$

It is easy to see that (4.3) is equivalent to the logistic equation (Devany, 1989)

$$u_{n+1} = 4bu_n(1-u_n), \qquad u_n = (1/2)(1-z_n) \tag{6}$$

However, it is important to realize that although (4.3) is independent of $x_n$ and $y_n$, the inclusion of (4.1) and (4.2) constrains the system. For example, although (4.3) has two fixed points, the $SU(2)/U(1)$ DS has a unique fixed point, namely $(0, 0, 1)$, i.e., $x_n = 0$, $y_n = 0$, and $z_n = 1$, for all $n = 1, 2, 3, \ldots$.

Stability analysis implies that the system is stable for $b < 0.25$. Numerical studies show that for $0.25 < b < 0.5$ the $x-y$ subsystem is chaotic, while $z$ is stable. The value $b = 0.5$ is a special point for the system and corresponds to $z = 0$ and a 4-cycle in the $x-y$ subsystem. For $0.5 < b \leq 1$ there is chaos in the $x-y$ subsystem, while $z$ is stable for $0.5 < b \leq 0.75$. For $0.75 < b \leq 0.87$ a 2-cycle forms, for $0.87 < b < 0.89$ a 4-cycle exists, for $0.89 < b < 0.9$ it becomes an 8-cycle, and for $0.9 < b < 1$ there is chaos. The trajectories are shown in Figs. 1–4.

Substituting $b = 1$ in system (4), we get

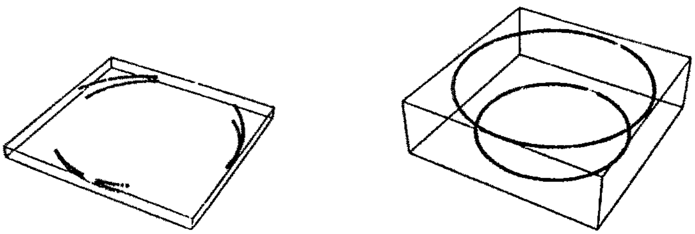
$$\frac{x_{n+1}}{y_{n+1}} = \frac{x_n}{y_n}$$

hence

$$x_n = ay_n \tag{7}$$

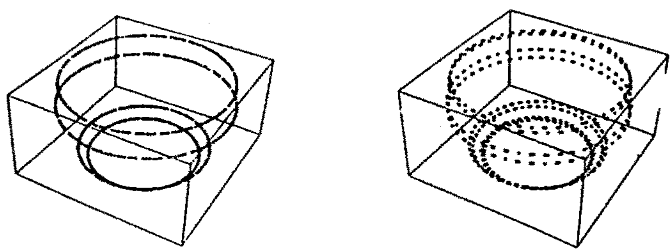
b = 0.25 b = 0.35

**Fig. 1.** Trajectories with $b = 0.25$ and $b = 0.35$.



b = 0.75 b = 0.8

**Fig. 2.** Trajectories with $b = 0.75$ and $b = 0.8$.



b = 0.88 b = 0.89

**Fig. 3.** Trajectories with $b = 0.88$ and $b = 0.89$.

where $a$ is a constant. The intersection of the sphere (7) with the sphere (3) is a circle. This explains the results in Fig. 4.

The value $b = 0.5$ is a special point for the system (4) and represents a discontinuity of its dynamical behavior in the parameter space since the system is chaotic for $0.25 < b < 0.5$ and for $0.5 < b \leq 1$. Setting $b = 0.5$ in (4.3), we get $z_n = 0$, $x_{n+1} = -y_n$, and $y_{n+1} = x_n$, which implies $x_{n+4} = x_n$, $y_{n+4} = y_n$, i.e., a 4-cycle. This explains the numerical results.

The sufficient condition that a dynamical system is chaotic is that it has a positive Lyapunov exponent (Wolf *et al.*, 1985). Since (4.3) is independent of $x_n$, $y_n$, one of the three Lyapunov exponents of the system

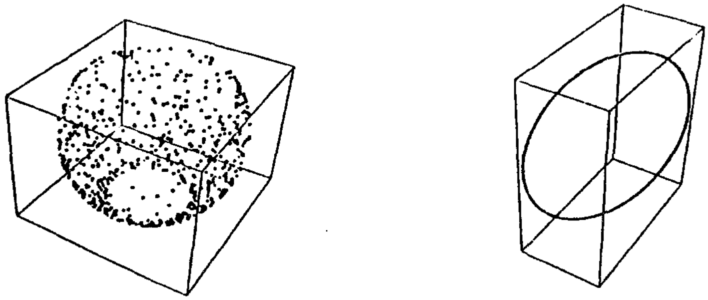



**Fig. 4.** Trajectories with $b = 0.92$ and $b = 1.0$.

(4) is equal to that of (4.3), which is known to be positive for $b > 0.9$. Therefore we have the following proposition:

*Proposition 1.* The system (4) is chaotic for $b > 0.9$.

## 3. THE $SU(2)/U(1)$ LINKS

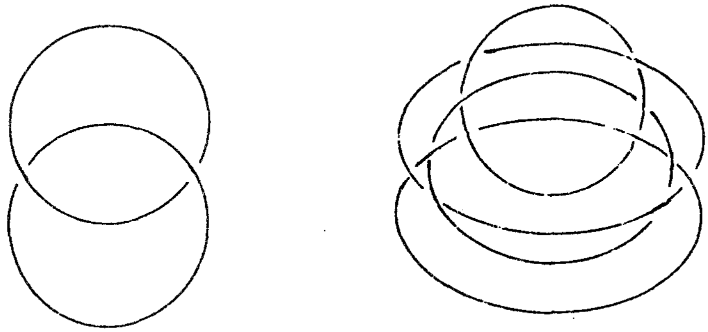
The trajectories of the chaotic dynamical system are a rich source for knots and links. Two examples are Lorenz (Birman and Williams, 1983) and Rossler (Ahmed and El-Rifai, 1992) knots. Here we define links corresponding to the cycles in the $z$-subsystem of the $SU(2)/U(1)$ DS. A standard problem in using trajectories obtained by numerical solutions to define knots and links is that in general these solutions are not very accurate in detecting closed paths. Here we do not have this problem since we know that the trajectories lie on a sphere (3). Also, the $z$ coordinate is fixed in the cycles and hence the trajectories lie in a closed circle. In Fig. 5 we show the links corresponding to the *2*-cycle and a *4*-cycle in the $z$ system.

We define the links by taking one of the crossings to be an up and the next one to be a down and so on. Changing the initial crossing will reduce the links. Since we consider only $2^n$-cycles $n = 1, 2, 3, \ldots$, the alternating up–down construction is consistent. We call the resulting links $SU(2)/U(1)$ links. Sarkovski's theorem (Devany, 1989) guarantees that the cycles, hence links, exist for all $n = 1, 2, 3, \ldots$. Our construction implies the following proposition.

*Proposition 2.*

2.1. All $SU(2)/U(1)$ links are not knots.

2.2. The $SU(2)/U(1)$ links corresponding to the $2^n$-cycles with $n = 1, 2$ are represented by elements of the braid group $B_2$, $B_6$ and are the closure of the following braids, respectively:



n = 1 n = 2

**Fig. 5.** $SU(2)/U(1)$ links for $n = 1$ and 2, respectively.

$$\begin{array}{ll} \sigma_1^2 & n=1 \\ (\sigma_1\sigma_3\sigma_5)(\sigma_2^{-1}\sigma_4^{-1}\sigma_5)(\sigma_5\sigma_2^{-1}\sigma_4^{-1})(\sigma_1\sigma_3\sigma_5) & n=2 \end{array} \tag{8}$$

2.3. The sets of the $SU(2)/U(1)$ links and Rossler knots are disjoint.

*Proof.*

2.1. The links consist of different closed curves, while knots contain a unique closed curve.

2.2. By construction.

2.3. Rossler knots are not links and using part 2.1 of this proposition, then they are disjoint.

*Conjectures.* 1. For $n > 1$ the $SU(2)/U(1)$ links are the closure of the following braids:

$$\begin{array}{l} (\sigma_1\sigma_3\cdots\sigma_{k-1})(\sigma_2^{-1}\sigma_4^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1})(\sigma_{k-1}\sigma_{k-2}^{-1}\sigma_{k-4}^{-1}\cdots\sigma_4^{-1}\sigma_2^{-1}) \\ (\sigma_1\sigma_3\cdots\sigma_{k-1}), \qquad k = 2^{n-1}(2^n - 1) \end{array} \tag{9}$$

2. For $n > 1$ the $SU(2)/U(1)$ links and Lorenz links are disjoint.

Finally, the rapid increase in the rank of the braid group corresponding to $SU(2)/U(1)$ links as $n$ increases ($B_2$ for $n = 1$, $B_6$ for $n = 2$, $B_{28}$ for $n = 3$, etc.) forces the need to use computer-aided methods to study the invariants of these links e.g., Jones polynomials (Jones, 1987). An effective program using *Mathematica* (Wolfram, 1990) has been designed by El-Misiery (1993).

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